Subelliptic boundary conditions for $\mathrm{Spin}_{\mathbb{C}}$–Dirac operators, gluing, relative indices, and tame Fredholm pairs

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Let X be a $\mathrm{Spin}_{\mathbb{C}}$ manifold with boundary, such that the $\mathrm{Spin}_{\mathbb{C}}$ structure is defined near the boundary by an almost complex structure, which is either strictly pseudoconvex or pseudoconcave (and hence contact). Using generalized Szegő projectors, we define modified $\bar\partial$-Neumann boundary conditions, $\mathcal{R}^{eo}$, for spinors, which lead to subelliptic Fredholm boundary value problems for the $\mathrm{Spin}_{\mathbb{C}}$–Dirac operator, $\eth^{eo}$. To study the index of these boundary value problems we introduce a generalization of Fredholm pairs to the "tame" category. In this context, we show that the index of the graph closure of $(\eth^{eo}, \mathcal{R}^{eo})$ equals the tame relative index, on the boundary, between $\mathcal{R}^{eo}$ and the Calderon projector. Let X_0 and X_1 be strictly pseudoconvex, $\mathrm{Spin}_{\mathbb{C}}$ manifolds, as above. Let $\phi: bX_1 \to bX_0$, be a contact diffeomorphism, S_0, S_1 denote generalized Szegő projectors on bX_0, bX_1, respectively, and $\mathcal{R}_0^{eo}$, $\mathcal{R}_1^{eo}$, the subelliptic boundary conditions they define. If $\overline{X_1}$ is the manifold X_1 with its orientation reversed, then the glued manifold $X = X_0 \amalg_\phi \overline{X_1}$ has a canonical $\mathrm{Spin}_{\mathbb{C}}$ structure and Dirac operator, $\eth_X^{eo}$. Applying these results we obtain a formula for the relative index, $\text{R-Ind}(S_0, \phi^* S_1)$,

$$\text{R-Ind}(S_0, \phi^*S_1) = \text{Ind}(\eth_X^e) - \text{Ind}(\eth_{X_0}^e, \mathcal{R}_0^e) + \text{Ind}(\eth_{X_1}^e, \mathcal{R}_1^e).$$

As a special case, this formula verifies a conjecture of Atiyah and Weinstein [(1997) *RIMS Kokyuroku* 1014:1–14] for the index of the quantization of a contact transformation between cosphere bundles.

index formula | contact manifold | Fourier integral operator | Atiyah–Weinstein conjecture | almost complex manifolds

The $\bar\partial$–Neumann problem has served as the technical foundation and a source of inspiration for research in several complex variables since the seminal work of Spencer, Kohn, and Nirenberg. In this article I describe a generalization of these boundary conditions to $\mathrm{Spin}_{\mathbb{C}}$ manifolds whose boundaries satisfy appropriate convexity conditions. The results demonstrate the remarkable robustness of the $\bar\partial$–Neumann condition, in that, properly understood, the analytic properties of these boundary value problems do not rely on the integrability of the almost complex structure.

The analytic details of these results require an amalgamation of the classical calculus of pseudodifferential operators with the Heisenberg calculus. This amalgamation, called the extended Heisenberg calculus, is described in refs. 1 and 2. The Heisenberg calculus is an algebra of pseudodifferential operators, canonically defined on any contact manifold, (Y, H), with $H \subset TY$, the contact structure. It is a filtered algebra of operators, acting between sections of vector bundles E, $F \to Y$. We let $\Psi_H^m(Y; E, F)$ denote the Heisenberg operators of order m, mapping $\mathcal{C}^\infty(Y; E)$ to $\mathcal{C}^\infty(Y; F)$. The Heisenberg calculus was introduced by, among others, Beals-Greiner, Taylor, and Dynin (see refs. 3 and 4).

Using this technology we reduce index computations for subelliptic boundary value problems for the $\mathrm{Spin}_{\mathbb{C}}$–Dirac operator to computations on the boundary itself. To do the analysis on the boundary we generalize the notion of a Fredholm pair to that of a tame Fredholm pair (see ref. 5), thus providing a unified framework in which to analyze both elliptic and subelliptic boundary value problems. This work has led to subelliptic, Fredholm boundary value problems for the $\mathrm{Spin}_{\mathbb{C}}$–Dirac operator, "gluing formulae" for holomorphic Euler characteristics and a very general index theorem that relates indices of Dirac operators, relative indices of generalized Szegő projectors, and indices of subelliptic boundary value problems. Using this index theorem, we settle a long outstanding conjecture of Atiyah and Weinstein and reduce the relative index conjecture in 3D-CR (Cauchy–Riemann) geometry to a conjecture of Ozbagci and Stipsicz in 3D contact geometry (see ref. 6). Complete details of these results can be found in refs. 7–9.

Subelliptic Boundary Value Problems for the $\mathrm{Spin}_{\mathbb{C}}$–Dirac Operator

The arena for this work is a $\mathrm{Spin}_{\mathbb{C}}$ manifold, X, with boundary. We usually assume that there is an almost complex structure, J, defined in a neighborhood of bX, and that the $\mathrm{Spin}_{\mathbb{C}}$ structure, near the boundary, is that defined by J. If the almost complex structure is defined on $U \subset X$, then $TX\restriction_U \otimes \mathbb{C}$ splits as $T^{1,0}X\restriction_U \oplus T^{0,1}X\restriction_U$. This, in turn, defines a splitting of the exterior algebra of $T^*X\restriction_U \otimes \mathbb{C}$ into (p, q) types: $\oplus_{p,q}\Lambda^{p,q}X\restriction_U$. In this case the bundle of spinors, $\$\restriction_U$, is canonically identified with $\oplus_q \Lambda^{0,q}X\restriction_U$. We denote the Clifford action of the one form η on a spinor σ by $c(\eta)\cdot\sigma$, and let $\nabla^{\$}$ denote a connection on the bundle of spinors, compatible with the Clifford action. If $\{V_j\}$ is a local framing for TX and $\{\eta_j\}$ the corresponding dual frame for T^*X, then locally the Dirac operator is given by

$$\eth\sigma = \sum_{j=1}^{2n} c(\eta_j)\cdot\nabla^{\$}_{V_j}\sigma. \quad [1]$$

If $E \to X$ is a complex vector bundle and ∇^E is a connection on E, then a compatible connection on $\$ \otimes E$ is given by

$$\nabla^{\$\otimes E} = \nabla^{\$} \otimes \mathrm{Id}_E + \mathrm{Id}_{\$} \otimes \nabla^E. \quad [2]$$

Using this connection in Eq. **1**, we obtain a twisted Dirac operator acting on sections of $\$ \otimes E$.

We assume that bX satisfies one of several natural convexity conditions, which are familiar from the complex case. If ρ is a defining function for bX, with $\rho < 0$ in X, then $\theta = -i\bar\partial\rho\restriction_{bX}$ is a real one form. The null space of this one form, $H \subset TbX$, is invariant under J, and bX is strictly pseudoconvex (concave) if the Levi form $\mathcal{L}(\cdot,\cdot) = d\theta(\cdot, J\cdot)$ is a positive (negative) definite form on $H \times H$. This implies that (bX, H) is a contact manifold.

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The choice of a contact form specifies a co-orientation. We call a $\text{Spin}_{\mathbb{C}}$ manifold with $\text{Spin}_{\mathbb{C}}$ structure defined near to bX by an almost complex structure, having a positive (negative) Levi form a strictly pseudoconvex (pseudoconcave) $\text{Spin}_{\mathbb{C}}$ manifold. If the Levi form is nondegenerate at every point of bX, and hence of fixed signature $(n - q, q - 1)$, for a $1 \leq q \leq n$, then an analogous discussion applies. For simplicity we focus on the strictly pseudoconvex/concave cases, $(q = 1/q = n)$.

The Holomorphic Case. We first consider the holomorphic case, letting X be a complex manifold with boundary. For a section $\omega \in \mathscr{C}^\infty(\bar{X}; \Lambda^{p,q} X)$, the $\bar{\partial}$–Neumann condition, is the requirement that $\bar{\partial}\rho \rfloor \omega \restriction_{bX} = 0$. Under appropriate convexity conditions, using this boundary condition to define the domain for the quadratic form

$$\mathfrak{Q}(\omega) = \langle \bar{\partial}\omega, \bar{\partial}\omega \rangle + \langle \bar{\partial}^*\omega, \bar{\partial}^*\omega \rangle, \quad [3]$$

the Friedrichs extension theorem can be used to define an unbounded self adjoint operator on $L^2(X; \Lambda^{p,q})$. This operator is denoted $\Box^{p,q}$ and is called the Kohn–Laplacian. If the complex dimension of X is n, then the condition $Z(q)$ is the requirement that the Levi form has at least $n - q$ positive eigenvalues, or at least $q + 1$ negative eigenvalues at each point of bX. If the manifold is strictly pseudoconvex, then it satisfies $Z(q)$ for $1 \leq q$. If X satisfies $Z(q)$, then $\Box^{p,q}$ is a self adjoint, Fredholm operator with a compact resolvent. If X satisfies $Z(q)$ for all q, but for a single q_0, lying between 0 and n, then $\Box^{p,q}$ is a self adjoint operator for all $0 \leq q \leq n$ and has a compact resolvent, provided that $q \neq q_0$. In general $\Box^{p,q_0}$ has closed range, but an infinite dimensional null space. For example, if X is strictly pseudoconvex then the null space of $\Box^{p,0}$ is infinite dimensional, whereas, if X is strictly pseudoconcave, then $\Box^{p,n-1}$ has an infinite dimensional null space.

The complex structure on X induces a CR structure on bX. We let $\Lambda_b^{p,q} bX$ denote the bundle of (p, q) b forms on bX, and $\bar{\partial}_b$ the canonical operator $\bar{\partial}_b : \mathscr{C}^\infty(bX; \Lambda_b^{p,q} bX) \to \mathscr{C}^\infty(bX; \Lambda_b^{p,q+1} bX)$, defined, for $q = 0$, by

$$\bar{\partial}_b f = df \restriction_{T_b^{0,1} bX}. \quad [4]$$

For these basic definitions see ref. 10. Formally the Kohn–Rossi Laplacian is given by

$$\Box_b^{p,q} = \bar{\partial}_b^* \bar{\partial}_b + \bar{\partial}_b \bar{\partial}_b^*. \quad [5]$$

If X satisfies $Z(q)$ and $Z(n - q - 1)$, then $\Box_b^{p,q}$ is an unbounded self adjoint operator with a compact resolvent. If bX is strictly pseudoconvex, then $\Box_b^{p,0}$ and $\Box_b^{p,n-1}$ have infinite dimensional null spaces. The null space of $\Box_b^{p,0}$ contains the boundary values of elements in the null space of $\Box^{p,0}$, as a finite codimension subspace. The null space of $\Box_b^{p,n-1}$ contains the boundary values of elements in the nullspace of $\Box^{p,n}$, with the dual $\bar{\partial}$-Neumann boundary condition, as a finite codimension subspace. The latter space is dual to the space of holomorphic $(n - p, 0)$ forms, which is again infinite dimensional.

We let $\mathcal{S}_p$ denote an orthogonal projection onto the null space of $\Box_b^{p,0}$, and $\bar{\mathcal{S}}_p$ denote an orthogonal projection onto the null space of $\Box_b^{p,n-1}$. By analogy with the case $p = 0$, we call $\mathcal{S}_p$ a Szegő projector and $\bar{\mathcal{S}}_p$ a conjugate Szegő projector. These are not classical pseudodifferential operators, but rather Heisenberg pseudodifferential operators, see ref. 3. In refs. 1 and 2 we define generalizations of these projectors, called generalized Szegő (or conjugate Szegő) projectors. Suppose that (Y, H) is a co-oriented contact manifold. If $E \to Y$ is a complex vector bundle, then a generalized Szegő projector S_E, acting on sections of E, is a projection operator in $\Psi_H^0(Y; E, E)$, which symbolically resembles a classical Szegő projector. In particular, the micro-support of its symbol lies in the positive contact direction. A generalized conjugate Szegő projector acting on sections of E is a projector in $\Psi_H^0(Y; E, E)$, resembling a conjugate Szegő projector, whose symbol has microsupport in the negative contact direction. Detailed definitions are given in ref. 1.

If X is a strictly pseudoconvex complex manifold, then the $\bar{\partial}$–Neumann condition for $q = 0$ does not define a Fredholm operator, as the infinite dimensional space of holomorphic sections of $\Lambda^{p,0}$ lies in the null space. Intuitively it is clear that if we modify the boundary condition to require that

$$\mathcal{S}_p(\sigma^{p,0} \restriction_{bX}) = 0, \quad [6]$$

then this should remove the null space. To get a formally self adjoint operator we also need to replace $[\bar{\partial}\rho \rfloor \bar{\partial}\sigma^{p,0}]_b = 0$ with $(\text{Id} - \mathcal{S}_p)[\bar{\partial}\rho \rfloor \bar{\partial}\sigma^{p,0}]_b = 0$. It is a consequence of the results below, that $\Box^{p,0}$, with these boundary conditions is a self adjoint operator with a compact resolvent. Indeed, it is not necessary to use the classical Szegő projector; in what follows we use generalized (conjugate) Szegő projectors to augment the $\bar{\partial}$–Neumann condition, and thereby obtain Fredholm operators.

The $\text{Spin}_{\mathbb{C}}$ Case. We now return to the case of a $\text{Spin}_{\mathbb{C}}$ manifold with boundary, X. As above, we assume that the $\text{Spin}_{\mathbb{C}}$ structure is defined in a neighborhood of the boundary by an almost complex structure. The $\text{Spin}_{\mathbb{C}}$ structure on X defines a bundle of complex spinors, $\mathcal{S}$, and a choice of hermitian metric on this bundle defines the $\text{Spin}_{\mathbb{C}}$–Dirac operator, $\eth$, as in ref. 1. In the part of X where the $\text{Spin}_{\mathbb{C}}$ structure is defined by the almost complex structure, there is a natural isomorphism between $\mathcal{S}$ and $\oplus_q \Lambda^{0,q} X$. If the complex structure is integrable and the Hermitian metric is Kähler, then $\eth = \bar{\partial} + \bar{\partial}^*$ (see refs. 11 and 12). If the complex structure is not integrable, then the Laplacian defined by the Dirac operator, $\eth^2$, does not preserve form degree. Hence it is more natural to work simultaneously with all form degrees, that is, with the bundle of spinors itself.

The $\bar{\partial}$–Neumann condition makes sense whenever the $\text{Spin}_{\mathbb{C}}$ structure is defined near the boundary by an almost complex structure: a complex spinor has a decomposition into $(0, q)$ forms: $\sigma = \sigma^{0,0} + \cdots + \sigma^{0,n}$. In this case, interior product of a complex spinor with $\bar{\partial}\rho$ is defined. For a strictly pseudoconvex $\text{Spin}_{\mathbb{C}}$ manifold, the modified $\bar{\partial}$–Neumann problem is defined on the $(0, q)$-form parts, for $q \geq 2$, by $\bar{\partial}\rho \rfloor \sigma^{0,q} \restriction_{bX} = 0$. To get a formally self adjoint, Fredholm problem for $\eth$ we need to modify the boundary condition in degree $(0, 0)$, and therefore in degree $(0, 1)$ as well. To that end we let S denote a generalized Szegő projector as defined in ref. 1. The conditions in degrees $(0, 0)$ and $(0, 1)$ are given by

$$S[\sigma^{0,0}]_b = 0 \text{ and } (\text{Id} - S)[\bar{\partial}\rho \rfloor \sigma^{0,1}]_b = 0. \quad [7]$$

We call this the modified $\bar{\partial}$–Neumann condition for a strictly pseudoconvex $\text{Spin}_{\mathbb{C}}$ manifold defined by the generalized Szegő projector S. This boundary condition is defined by a projector $\mathcal{R}_+$ acting on sections of $\mathcal{S} \restriction_{bX}$. Using the Clifford action of the orientation class, the bundle $\mathcal{S}$ and these operators can be split into even and odd parts: $\mathcal{S}^{eo}$, $\eth^{eo}$, $\mathcal{R}_+^{eo}$, respectively. If $\mathcal{B}^{eo}$ denotes such a projector acting on sections of $\mathcal{S}^{eo} \restriction_{bX}$, then the pairs $(\eth^{eo}, \mathcal{B}^{eo})$ denote the operators $\eth^{eo}$ acting on smooth sections, σ of $\mathcal{S}^{eo}$, which satisfy $\mathcal{B}^{eo}[\sigma \restriction_{bX}] = 0$.

There is an analogous construction of a modified $\bar{\partial}$–Neumann problem for a pseudoconcave manifold: Let $\bar{S}$ denote a generalized conjugate Szegő projector. We use the classical $\bar{\partial}$–Neumann condition in degrees $0 \leq q \leq n - 2$. In degree $(0, n - 1)$ we require that $\bar{S}[\sigma^{0,n-1}]_b = 0$ and in degree $(0, n)$ we have $(\text{Id} - \bar{S})[\bar{\partial}\rho \rfloor \sigma^{0,n}]_b = 0$. In this case, we denote the associated projectors by $\mathcal{R}_-^{eo}$. The operators $(\eth_\pm^e, \mathcal{R}_\pm^e)$ are formally adjoint to $(\eth_\pm^o, \mathcal{R}_\pm^o)$, and vice versa. As noted above, one can define an

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analogous boundary condition whenever the Levi form is nondegenerate, and hence of constant signature.

Our main analytic results concerning the modified $\bar{\partial}$–Neumann boundary conditions are summarized in the following theorem.

Theorem 1. *Suppose that X is a strictly pseudoconvex (pseudoconcave)* $\mathrm{Spin}_{\mathbb{C}}$ *manifold, and S ($\bar{S}$) is a generalized Szegő (conjugate Szegő) projector defining the modified $\bar{\partial}$–Neumann condition* $\mathcal{R}_+(\mathcal{R}_-)$. The operators, $\overline{(\eth^{eo}_{\pm},\ \mathcal{R}^{eo}_{\pm})}$, *obtained by graph closure of* $(\eth^{eo}_{\pm},\ \mathcal{R}^{eo}_{\pm})$, *are Fredholm and*

$$(\eth^{eo}_{\pm},\ \mathcal{R}^{eo}_{\pm})^* = \overline{(\eth^{oe}_{\pm},\ \mathcal{R}^{oe}_{\pm})}. \qquad [8]$$

Proof: This theorem is proved by observing that if σ is an L^2 section of $\mathcal{S}$ such that $\eth\sigma$ is also L^2, then $\sigma\restriction_{bX}$ is well defined as an element of $H^{-1/2}(bX;\ \mathcal{S})$. As the boundary conditions are defined by pseudodifferential projections, the requirement $\mathcal{R}_{\pm}[\sigma\restriction_{bX}] = 0$ makes distributional sense. Connected to the Dirac operators, $\eth^{eo}_{\pm}$, are the Calderon projectors, $\mathcal{P}^{eo}_{\pm}$. These are projections, acting on sections of $\mathcal{S}\restriction_{bX}$, with range equal to the boundary values of elements in the null spaces of $\eth^{eo}_{\pm}$. Seeley (13) proved that they are classical pseudodifferential operators of order 0. To proceed we consider the comparison operators:

$$\mathcal{T}^{eo}_{\pm} = \mathcal{R}^{eo}_{\pm}\mathcal{P}^{eo}_{\pm} + (\mathrm{Id} - \mathcal{R}^{eo}_{\pm})(\mathrm{Id} - \mathcal{P}^{eo}_{\pm}). \qquad [9]$$

Loosely speaking this allows us to compare the projectors $\mathcal{R}^{eo}_{\pm}$ that define the modified $\bar{\partial}$–Neumann boundary conditions with the Calderon projectors. The comparison operators are elements of the extended Heisenberg calculus. They are not elliptic is a classical sense, but rather in a graded, extended Heisenberg sense, and have a rather complicated symbol in the contact directions. Nonetheless, using symbolic computations within the extended Heisenberg calculus, we prove the following lemma:

Lemma 1. *If X is a strictly pseudoconvex (pseudoconcave)* $\mathrm{Spin}_{\mathbb{C}}$ *manifold, then there exist extended Heisenberg operators, $\mathcal{U}^{eo}_{\pm}$ so that*

$$\mathcal{U}^{eo}_{\pm}\mathcal{T}^{eo}_{\pm} = \mathrm{Id} - K^{eo}_{1\pm} \quad \mathcal{T}^{eo}_{\pm}\mathcal{U}^{eo}_{\pm} = \mathrm{Id} - K^{eo}_{2\pm}, \qquad [10]$$

where $K^{eo}_{1\pm}$, $K^{eo}_{2\pm}$ are smoothing operators.

Using this lemma, and the fact that, for $s \in \mathbb{R}$, $\mathcal{U}^{eo}_{\pm} : H^s \to H^{s-1/2}$ are bounded, we can apply a standard argument, using boundary layers, to prove the higher norm estimates:

Lemma 2. *If X is a strictly pseudoconvex (pseudoconcave)* $\mathrm{Spin}_{\mathbb{C}}$ *manifold, then for each $s \geq 0$, there is a positive constant C_s, so that if u is an L^2-solution to:*

$$\eth^{eo}_{\pm}u = f \in H^s(X) \text{ and } \mathcal{R}^{eo}_{\pm}[u\restriction_{bX}] = 0, \qquad [11]$$

in the sense of distributions, then $u \in H^{s+1/2}(X)$, and

$$\|u\|_{H^{s+1/2}(X)} \leq C_s[\|f\|_{H^s(X)} + \|u\|_{L^2(X)}]. \qquad [12]$$

Using a standard duality argument, the proof of the theorem is immediate from *Lemma 2.*

These analytic results easily extend to the case where the spin bundle is twisted with a complex vector bundle $E \to X$, and the Dirac operator is extended to act on sections of $\mathcal{S} \otimes E$. The classical $\bar{\partial}$–Neumann condition extends trivially to this case. We use generalized Szegő projectors (conjugate Szegő projectors) that act on sections of E to define the modifications to the $\bar{\partial}$–Neumann condition described above. In the sequel, we use the analytic results in this generality without further comment. When we want to emphasize that the spin bundle and Dirac operator have been so twisted, we use the notation $(\eth^{eo}_{E\pm},\ \mathcal{R}^{eo}_{E\pm})$. If $E = \Lambda^{p,0}X$, then we use the notation $(\eth^{eo}_{p\pm},\ \mathcal{R}^{eo}_{p\pm})$. If σ is a spinor defined on bX, satisfying $\mathcal{R}^{eo}_{\pm}\sigma = 0$, then the Hodge star takes σ to a spinor satisfying.

$$(\mathrm{Id} - \mathcal{R}^{eo}_{\mp})[{*\sigma}] = 0. \qquad [13]$$

Hence the projectors $(\mathrm{Id} - \mathcal{R}^{eo}_{-})$ $((\mathrm{Id} - \mathcal{R}^{eo}_{+}))$ define modified dual-$\bar{\partial}$–Neumann problems, which are Fredholm on a strictly pseudoconvex (pseudoconcave) $\mathrm{Spin}_{\mathbb{C}}$ manifold.

Gluing Formulae in the Holomorphic Case

The most immediate application of these results is to prove gluing formulae in the classical holomorphic case. Suppose that X is Kähler and strictly pseudoconvex, $V \to X$ is a holomorphic vector bundle, $\mathcal{S}_V$ is the classical Szegő projector onto the null space of $\bar{\partial}_b$ acting on sections of $V\restriction_{bX}$, and $\mathcal{R}^{eo}_{V+}$ the associated modified $\bar{\partial}$–Neumann boundary condition. In this case $\eth^2$ does preserve form degrees and we can compute the index of the Fredholm operator $(\eth^{e}_{V+},\ \mathcal{R}^{e}_{V+})$:

$$\mathrm{Ind}(\eth^{e}_{V+},\ \mathcal{R}^{e}_{V+}) = -\dim E^{V,1}_0 + \sum_{q=1}^{n} \dim H^q(X;\ V)(-1)^q. \qquad [14]$$

Here $E^{V,1}_0$ are the obstructions to extending $\bar{\partial}_b$ closed sections of $V\restriction_{bX}$ as holomorphic sections defined on X. This formula is established by identifying the null space of $(\eth^{eo}_{V+},\ \mathcal{R}^{eo}_{V+})$, with the $\bar{\partial}$–Neumann harmonic forms in degrees >1. In degree 1, a careful analysis shows that the null space of $\eth$ consists of the $\bar{\partial}$–Neumann harmonic V-valued (0, 1) forms, along with the finite dimensional vector space:

$$E^{V,1}_0 = \frac{\{\bar{\partial}\alpha : \alpha \in \mathcal{C}^{\infty}(\bar{X};\ V) \text{ and } \bar{\partial}_b\alpha_b = 0\}}{\{\bar{\alpha} : \alpha \in \mathcal{C}^{\infty}(\bar{X};\ V) \text{ and } \alpha_b = 0\}}. \qquad [15]$$

These results use an analogue of the Hodge theorem for the operators defined by the modified $\bar{\partial}$–Neumann conditions and lead to gluing formulae for holomorphic Euler characteristics.

Theorem 2. *Suppose that X is a compact Kähler manifold of dimension n, with a separating, strictly pseudoconvex hypersurface. $Y \to X$. Let $X_{\pm}$ denote the components of $X \setminus Y$. For each $0 \leq p \leq n$, we have*

$$\sum_{q=0}^{n} \dim H^{p,q}(X)(-1)^q = \mathrm{Ind}(\eth^{e}_{p+},\ \mathcal{R}^{e}_{p+}) + \mathrm{Ind}(\eth^{e}_{p-},\ \mathrm{Id} - \mathcal{R}^{e}_{p+}) \qquad [16]$$

$$\sum_{q=0}^{n} \dim H^{p,q}(X)(-1)^q = \mathrm{Ind}(\eth^{e}_{p+},\ \mathcal{R}^{e}_{p+}) + \mathrm{Ind}(\eth^{e}_{p-},\ \mathcal{R}^{e}_{p-}) - \sum_{q=1}^{n-2} \dim H^{p,q}_b(Y)(-1)^q.$$

The sum over $H^{p,q}_b$ is absent if $n = 2$.

Proof: This theorem is a consequence of the identification, in the Kähler case, of the null spaces of $(\eth^{eo}_{p\pm},\ \mathcal{R}^{eo}_{p\pm})$ with $\bar{\partial}$–Neumann harmonic forms along with modifications of some long exact sequences of Andreotti and Hill (14). The sequences appearing in that classic paper relate various sheaf cohomology groups of X_+, X_- and X. In each sequence there are several infinite dimensional groups, which precludes the direct computation of Euler characteristics. We modify these sequences, replacing the infinite dimensional groups with finite dimensional groups. Taking Euler characteristics then leads easily to Eq. **16**.

Relative Indices and Tame Fredholm Pairs

Let H be a Hilbert space and H_1, H_2 closed subspaces of H. If $H_1 \cap H_2$ is finite dimensional and $H_1 + H_2$ is closed and finite codimensional, then we say that (H_1, H_2) is a Fredholm pair. The index of the pair is defined to be:

$$\mathrm{Ind}(H_1, H_2) = \dim(H_1 \cap H_2) - \dim(H/[H_1 + H_2]). \quad [17]$$

Let P_1 be the orthogonal projection onto H_1 and P_2 the orthogonal projection onto $H_2^{\perp}$. The condition that (H_1, H_2) be a Fredholm pair is equivalent to the condition that

$$P_1 : H_2^{\perp} \rightarrow H_1 \quad [18]$$

be a Fredholm operator. The index of this operator is called the relative index of the pair (P_1, P_2) and is denoted by R-Ind(P_1, P_2). A simple calculation shows that

$$\text{R-Ind}(P_1, P_2) = \mathrm{Ind}(H_1, H_2). \quad [19]$$

Roughly speaking, the relative index measures the "dimension" of the formal difference $\mathrm{Im}\, P_1 - \mathrm{Im}\, P_2$. The notion of Fredholm pair plays a central role in the index theory of Dirac operators on manifolds with boundary, using Atiyah–Patodi–Singer-type boundary conditions (see ref. 5). In this context, the projectors are classical pseudodifferential operators, which usually have the same principal symbol as the Calderon projector. For example if P_1, and P_2, are two such projectors, then the Agranovich–Dynin theorem states that

$$\mathrm{Ind}(\eth^{e}, P_1) - \mathrm{Ind}(\eth^{e}, P_2) = \text{R-Ind}(P_1, P_2). \quad [20]$$

If $P_2 = \mathcal{P}^{e}$ is the Calderon projector, then $\mathrm{Ind}(\eth^{e}, \mathcal{P}^{e}) = 0$ and so

$$\mathrm{Ind}(\eth^{e}, P_1) = \text{R-Ind}(P_1, \mathcal{P}^{e}). \quad [21]$$

For nonlocal boundary value problems, with the boundary condition defined by a projector, the index is not, in general, determined by the complete symbol of the projector. In the classical Atiyd–Patodi–Singer case, the connected components of the space of projectors defining elliptic boundary conditions are labeled by $\mathrm{Ind}(\eth^{e}, P)$, and hence by R-Ind$(P, \mathcal{P}^{e})$ (see ref. 5).

The pair $(\mathrm{Im}\, \mathcal{R}_{+}^{e}, \mathrm{Im}(\mathrm{Id} - \mathcal{P}_{+}^{e}))$ is not a Fredholm pair, as the sum of these two subspaces is not closed. We have therefore generalized this circle of ideas to allow for boundary conditions, like the modified $\bar{\partial}$–Neumann condition. These are called tame Fredholm pairs. The principal consequences of the theory of tame Fredholm pairs, which is outlined in the final section, are summarized in the following theorem.

Theorem 3. *Suppose that X is a* Spin$_{\mathbb{C}}$ *manifold with strictly pseudoconvex boundary. Let S be a generalized Szegő projector, defining a modified $\bar{\partial}$–Neumann boundary condition $\mathcal{R}_{+}^{eo}$. If $\mathcal{P}_{+}^{eo}$ are the Calderon projectors defined by $\eth^{eo}$, then $(\mathcal{P}_{+}^{eo}, \mathcal{R}_{+}^{eo})$ are* tame Fredholm pairs, *moreover*.

$$\mathrm{Ind}(\eth_{+}^{eo}, \mathcal{R}_{+}^{eo}) = \text{R-Ind}(\mathcal{P}_{+}^{eo}, \mathcal{R}_{+}^{eo}). \quad [22]$$

Remark. The relative index on the right side of Eq. **22** is of a tame Fredholm pair.

Proof: To prove the theorem we first identify the null space of $(\eth_{+}^{eo}, \mathcal{R}_{+}^{eo})$ with the null space of $\mathcal{R}_{+}^{eo}\restriction_{\mathrm{Im}\, \mathcal{P}_{+}^{eo}}$. Next, we use the facts that $(\eth_{+}^{eo}, \mathcal{R}_{+}^{eo})^{*} = \overline{(\eth_{+}^{oe}, \mathcal{R}_{+}^{oe})}$ and that $\mathcal{P}_{+}^{eo*} = \boldsymbol{c}(dt)(\mathrm{Id} - \mathcal{P}_{+}^{oe})\boldsymbol{c}(dt)^{-1}$ to identify the cokernels.

The theorem states that the formula for the index of an Atiyah-Patodi-Singer-type boundary value problem given in Eq. **21** extends to boundary conditions defined by modifying the $\bar{\partial}$–Neumann condition. This formula indicates that the tame Fredholm formalism should provide an entirely unified treatment of a broad range of boundary conditions.

In refs. 1, 15, 16, and 28 it is shown that if (Y, H) is a compact contact manifold and S_1, S_2 are generalized Szegő projectors, then $S_2 : \mathrm{Im}\, S_1 \rightarrow \mathrm{Im}\, S_2$ is a Fredholm operator. We denote its index by R-Ind(S_1, S_2). We first obtain a generalization of the Agranovich–Dynin formula.

Theorem 4. *Suppose that X is a strictly pseudoconvex* Spin$_{\mathbb{C}}$ *manifold. Let S_1 and S_2 be generalized Szegő projectors defined on bX. If $\mathcal{R}_{+1}$, $\mathcal{R}_{+2}$ are the modified $\bar{\partial}$–Neumann conditions defined by these projectors then*

$$\text{R-Ind}(S_1, S_2) = \mathrm{Ind}(\eth_{+}^{e}, \mathcal{R}_{+2}^{e}) - \mathrm{Ind}(\eth_{+}^{e}, \mathcal{R}_{+1}^{e}). \quad [23]$$

Proof: This theorem follows from *Theorem 3* and the cocycle formula for tame Fredholm pairs applied to the right side of

$$\mathrm{Ind}(\eth_{+}^{eo}, \mathcal{R}_{+2}^{eo}) - \mathrm{Ind}(\eth_{+}^{eo}, \mathcal{R}_{+1}^{eo})$$
$$= \text{R-Ind}(\mathcal{P}_{+}^{eo}, \mathcal{R}_{+2}^{eo}) - \text{R-Ind}(\mathcal{P}_{+}^{eo}, \mathcal{R}_{+1}^{eo}). \quad [24]$$

Let X_1, X_2 be strictly pseudoconvex Spin$_{\mathbb{C}}$ manifolds and suppose that $\phi : bX_1 \rightarrow bX_2$, is a contact diffeomorphism of the boundary. Adapting the "invertible double" construction from ref. 5, we construct a canonical Spin$_{\mathbb{C}}$ structure on the compact manifold $X_{12} \simeq X_1 \amalg_{\phi} \overline{X_2}$. Here $\overline{X_2}$ denotes X_2 with its orientation reversed. If ν is a 1-form conormal to $bX_1 \simeq bX_2$, then along the common boundary, we use $\boldsymbol{c}(\nu)$ to glue even/odd spinors on X_1 to odd/even spinors on $\overline{X_2}$. Using formula **23** we deduce a formula for the relative index of two Szegő projectors conjectured by Atiyah and Weinstein (6).

Theorem 5. *If S_1 and S_2 are generalized Szegő projectors, defined on bX_1, bX_2, respectively, then*

$$\text{R-Ind}(S_1, \phi^{*}S_2) = \mathrm{Ind}(\eth_{X_{12}}^{e}) - \mathrm{Ind}(\eth_{X_1}^{e}, \mathcal{R}_{+1}^{e})$$
$$+ \mathrm{Ind}(\eth_{X_2}^{e}, \mathcal{R}_{+2}^{e}). \quad [25]$$

There many interesting special cases of this formula. For example, if X_1, and X_2 are Stein manifolds and S_1, S_2 are the classical Szegő projectors, then formula **14** implies that the boundary terms in Eq. **25** vanish, that is

$$\text{R-Ind}(S_1, \phi^{*}S_2) = \mathrm{Ind}(\eth_{X_{12}}^{e}). \quad [26]$$

The classical Atiyah–Singer theorem provides a cohomological formula for the right side of Eq. **26**. If X_1 and X_2 are equivalent, as Spin$_{\mathbb{C}}$ manifolds, then $\mathrm{Ind}(\eth_{X_{12}}^{e}) = 0$.

Suppose the $X_j = B^{*}M_j$, $j = 1, 2$ are the coball bundles of compact manifolds, M_j, and that $\phi : S^{*}M_2 \rightarrow S^{*}M_1$ is a contact diffeomorphism of their boundaries. Formula **26** gives the index of a Fourier integral operator $F_{\phi} : \mathscr{C}^{\infty}(M_1) \rightarrow \mathscr{C}^{\infty}(M_2)$, obtained by "quantizing" ϕ (see ref. 6). As the coball bundles have natural complex structures with respect to which they are Stein manifolds, we obtain that $\mathrm{Ind}(F_{\phi}) = \mathrm{Ind}(\eth_{X_{12}}^{e})$. This case is essentially the case explicitly considered in the Atiyah–Weinstein conjecture. A similar formula was given in ref. 17, and in a special case in ref. 1. This result can be combined with the solution of the Boutet de Monvel–Guillemin conjecture, which states that fiber integration defines an isomorphism between the holomorphic $(n, 0)$-forms on $B^{*}M$ (in its natural complex structure) and smooth functions on M, to give a completely analytic proof that the index of a elliptic pseudodifferential operator on a compact manifold M equals the index of a Dirac operator on $B^*M \amalg \overline{B^*M}$.

A rather different application concerns the problem of embeddability for CR structures defined on compact three-manifolds. Suppose that (Y, H) is a compact 3D, contact manifold. A complex structure on the fibers of H defines a strictly pseudoconvex CR structure on Y, and hence a splitting: $H \otimes \mathbb{C} = H^{1,0} \oplus H^{0,1}$. The $\bar{\partial}_b$ operator is defined by $\bar{\partial}_b f = df \restriction_{H^{0,1}}$. The CR structure is embeddable if the null space of $\bar{\partial}_b$ contains an embedding of Y into $\mathbb{C}^N$ for some N. From work of Kohn (18), Harvey and Lawson (19), and de Monvel (20), it follows that this requirement is equivalent to the assertion that $(Y, H^{0,1})$ arises as the boundary of compact, complex manifold with boundary. In three dimensions, this property is extremely unstable under small deformations of the CR structure see refs. 15, 16, 21–23, and 28.

Let $\mathcal{S}_1$ denote the Szegő projector for a reference, embeddable CR structure on (Y, H), and $\mathcal{S}_2$ the projector onto ker $\bar{\partial}_b$ for a deformation of this CR structure. The deformed structure is embeddable if and only if $\mathcal{S}_2 : \operatorname{Im} \mathcal{S}_1 \to \operatorname{Im} \mathcal{S}_2$ is a Fredholm operator. In ref. 15 and 28 it is conjectured that R-Ind$(\mathcal{S}_1, \mathcal{S}_2)$ only assumes finitely many values among small embeddable deformations of a reference embeddable structure. This would imply that the set of small embeddable deformations of the CR structure is closed in the $\mathscr{C}^\infty$ topology. Using the classical Atiyah–Singer theorem to evaluate Ind$(\eth^{e}_{X_{12}})$, formula **25** gives a very explicit formula for this relative index.

Theorem 6. *Suppose that (Y, H) is a compact three dimensional, contact manifold, with two embeddable CR structures, $H_1^{0,1}$, $H_2^{0,1}$. Suppose that X_1, X_2 are smooth complex surfaces with $bX_j = (Y, H_j^{0,1})$, $j = 1, 2$. The relative index is given by the following formula:*

$$\text{R-Ind}(\mathcal{S}_1, \mathcal{S}_2) = \dim H^{0,1}(X_1) - \dim H^{0,1}(X_2) + \frac{\text{sig}[X_1] - \text{sig}[X_2] + \chi[X_1] - \chi[X_2]}{4}. \qquad [27]$$

The sig$[X_j]$, $\chi[X_j]$ *are, respectively, the signature of the intersection form, and Euler characteristic of X_j.*

One can show that, for sufficiently small embeddable deformations of the reference CR structure, the relative index, R-Ind$(\mathcal{S}_1, \mathcal{S}_2)$ is non-negative, and therefore this formula makes apparent that the subtlety of the relative index conjecture lies in the fact that the topology of the filling surface can change. Indeed, it is now known that there are contact three manifolds with fillable CR structures bounding infinitely many topologically distinct Stein surfaces. On the other hand, Ozbagci and Stipsicz (24, 25) have conjectured that, among Stein surfaces bounding a given contact 3-manifold, the numbers sig$[X]$, $\chi[X]$ assume only finitely many values. The truth of this conjecture would, of course, imply a strengthened, global form of the CR-relative index conjecture. Both have been proved in many interesting cases (see refs. 24–26).

Tame Fredholm Pairs

In ref. 9 I set up a very general functional analytic framework, which I call the "tame" category. Our usage of the term tame is very similar, in spirit, to its usage in connection with the Nash–Moser implicit function theorem. Briefly we have a nested family of Hilbert spaces $\{(H_s, \langle\cdot,\cdot\rangle_s) : s \in \mathbb{R}\}$, such that $H_\infty = \cap_{s\in\mathbb{R}} H_s$, the "smooth elements," and $H_{-\infty} = \cup_{s\in\mathbb{R}} H_s$, the "distributions," are naturally dual. The inner product on H_0 extends to a pairing of $H_s \times H_{-s}$, establishing isomorphisms $H_s' \simeq H_{-s}$, for all $s \in \mathbb{R}$.

An operator $A : H_\infty \to H_{-\infty}$ is tame if there is a k such that $A : H_s \to H_{s-k}$ is bounded for every $s \in \mathbb{R}$. In this case A is an operator of order (at most) k. The operators $K : H_{-\infty} \to H_\infty$, are the smoothing operators. We assume that if K is a smoothing operator, then, for any $s \in \mathbb{R}$, $K : H_s \to H_s$ is trace class. An operator, A, is tamely elliptic if there is a tame operator B such that Id $- AB$, and Id $- BA$ are smoothing. B is called a tame parametrix for A. We assume, finally, that if T is of nonpositive order, then Id $+ T^*T$ is elliptic.

To prove the results described in *Relative Indices and Tame Fredholm Pairs*, we give an extension of the theory of Fredholm pairs to the tame category.

Definition: We say that a pair of tame projectors (P, Q) of order 0 is a tame Fredholm pair if the comparison operator $T = PQ + (\text{Id} - P)(\text{Id} - Q)$ has a tame parametrix, U.

The parametrix satisfies

$$TU = \text{Id} - K_1 \quad UT = \text{Id} - K_2, \qquad [28]$$

with K_1, K_2 smoothing operators. In the classical theory of Fredholm pairs, the operator U would be a bounded operator. The classical theory of elliptic boundary value problems can be rephrased in this language and analyzed via the properties of the comparison operator, T. In the tame (nonclassical) case, U is typically of positive order. We define a Hilbert space, $H_U \subset H_0$, as the closure of H_∞ in the norm:

$$\|x\|_U^2 = \langle x, x\rangle_0 + \langle Ux, Ux\rangle_0. \qquad [29]$$

If L is a Hilbert space, then we use tr_L to denote the trace functional on trace class operators $A : L \to L$.

Theorem 7. *Let (P, Q) be a tame Fredholm pair, then*

$$QP : H_0 \cap \text{range } P \to H_U \cap \text{range } Q, \qquad [30]$$

is a Fredholm operator. The index of this operator, which defines the relative index, R-Ind(P, Q), *is given by*

$$\text{R-Ind}(P, Q) = \text{tr}_{H_0}(PK_2P) - \text{tr}_{H_0}(QK_1Q). \qquad [31]$$

Proof: The proof of most of this theorem uses only elementary Hilbert space theory. This leads to a trace formula for the relative index of a slightly different form:

$$\text{R-Ind}(P, Q) = \text{tr}_{H_0}(PK_2P) - \text{tr}_{H_U}(QK_1Q). \qquad [32]$$

The remarkable invariance properties of the trace, and the fact that Id $+ T^*T$ is elliptic allow one to show that

$$\text{tr}_{H_U}(QK_1Q) = \text{tr}_{H_0}(QK_1Q),$$

and hence prove Eq. **31**.

In our applications of this functional analytic framework, we use the L^2-Sobolev spaces, $\{H^s(bX_\pm; E \otimes \mathcal{S}^{eo}) : s \in \mathbb{R}\}$ as the ladder of Hilbert spaces. The tame operators include the extended Heisenberg operators. If the operator T is elliptic in the graded, extended Heisenberg sense, then the projectors (P, Q) define a tame Fredholm pair. In this case we can use Lidskii's theorem to express the traces in Eq. **31** as integrals, over bX, of smooth Schwartz kernels restricted to the diagonal (see ref. 27). This enormously facilitates the study of the index of the modified $\bar{\partial}$–Neumann problem.

Indeed, we can state very general versions of many of our results. Suppose that X is a Spin$_\mathbb{C}$ manifoid with boundary. We do not require the boundary to have a contact structure. Let $\eth^{eo}$ denote the (chiral) Spin$_\mathbb{C}$–Dirac operators and $\mathcal{P}^{eo}$ the Calderon projectors. Let $\mathcal{B}^{eo}$ be a pair of tame, self adjoint projectors acting on $\mathcal{S}^{eo} \restriction_{bX}$. If dt is a unit covector orthogonal to T^*bX, then these projectors define formally adjoint boundary conditions provided that

$$\mathcal{R}^{eo} = c(dt)(\mathrm{Id} - \mathcal{R}^{oe})c(dt)^{-1}. \quad [33]$$

As above we set $\mathcal{T}^{eo} = \mathcal{R}^{eo}\mathcal{P}^{eo} + (\mathrm{Id} - \mathcal{R}^{eo})(\mathrm{Id} - \mathcal{P}^{eo})$. Suppose that $(\mathcal{P}^{eo}, \mathcal{R}^{eo})$ is a tame Fredholm pair with respect to the L^2-Sobolev spaces of bX, with parametrix $\mathcal{U}^{eo}$. Below we say that a tame operator A, which maps H^s to H^{s-a} for all $s \in \mathbb{R}$, has Sobolev order a. As we define the order in terms of mapping properties, if A is a tame operator with Sobolev order α and B, a tame operator with Sobolev order β, then $A \circ B$ and $B \circ A$ are tame operators with Sobolev order $\alpha + \beta$.

Theorem 8. *Under the hypotheses above, suppose that $\mathcal{U}^{eo}$ have Sobolev order $1 - \alpha$. If $0 \leq \alpha$, then the graph closures of $(\eth^{eo}, \mathcal{R}^{eo})$ are Fredholm operators. If $0 < \alpha$, then these operators have compact resolvents and*

$$(\eth^{eo}, \mathcal{R}^{eo})^* = \overline{(\eth^{oe}, \mathcal{R}^{oe})}. \quad [34]$$

Proof: The proof of the first statement closely follows the proof of *Theorem 2* in ref. 8. Let Q^{eo} denote the fundamental solutions and $\mathcal{K}^{eo}$ the Poisson operators for $\eth^{eo}$. The only point that requires attention is that $f \mapsto \mathcal{K}^{eo}\mathcal{U}^{eo}\mathcal{R}^{eo}Q^{eo}f$ defines a bounded map from $L^2(X; \mathcal{S}^{eo})$ to itself. This is immediate from the well known mapping properties of $\mathcal{K}^{eo}$ and Q^{eo}, as well as the assumption that $\mathcal{R}^{eo}$ is tame with Sobolev order 0, and $\mathcal{U}^{eo}$ is tame with Sobolev order at most 1. Otherwise the proof of theorem 2 in ref. 8 applies verbatim to establish that the graph closures of $(\eth^{eo}, \mathcal{R}^{eo})$ are Fredholm. The domains of these operators are contained in H^{α}, which shows that the resolvents are compact if $\alpha > 0$.

The proof of Eq. **34** closely follows the proof of theorem 4 in ref. 8. In that argument we consider the system of Dirac operators:

$$\mathfrak{D}^{eo}_{\mu} = \begin{pmatrix} \eth^{oe} & -\mu \\ \mu & \eth^{eo} \end{pmatrix}. \quad [35]$$

Let $\mathcal{P}^{eo}_{\mu}$ denote the Calderon projectors for $\mathfrak{D}^{eo}_{\mu}$. If

$$\mathcal{R}^{eo}_{(2)} = \begin{pmatrix} \mathcal{R}^{oe} & 0 \\ 0 & \mathcal{R}^{eo} \end{pmatrix}, \quad [36]$$

and $\mathcal{T}^{eo}_{\mu} = \mathcal{R}^{eo}_{(2)}\mathcal{P}^{eo}_{\mu} + (\mathrm{Id} - \mathcal{R}^{eo}_{(2)})(\mathrm{Id} - \mathcal{P}^{eo}_{\mu})$, then the assumption that $\alpha > 0$ implies that

$$\mathcal{U}^{eo}_{0} = \begin{pmatrix} \mathcal{U}^{oe} & 0 \\ 0 & \mathcal{U}^{eo} \end{pmatrix} \quad [37]$$

inverts $\mathcal{T}^{eo}_{\mu}$ up to an error of negative order. For each $j \in \mathbb{N}$, using a finite Neumann series, we can get operators $\mathcal{U}^{eo}_{\mu j}$ so that $\mathcal{U}^{eo}_{\mu j}\mathcal{T}^{eo}_{\mu} - \mathrm{Id}$, $\mathcal{T}^{eo}_{\mu}\mathcal{U}^{eo}_{\mu j} - \mathrm{Id}$, have tame Sobolev order $-j$. This suffices to use the argument in the proof of *Theorem 4* to establish Eq. **34**.

We can now establish the basic relative index formula.

Theorem 9. *Assume that X is a* $\mathrm{Spin}_{\mathbb{C}}$ *manifold with boundary and $\mathcal{R}^{eo}$ are self adjoint projections acting on $\mathcal{S}^{eo}\restriction_{bX}$ satisfying Eq.* **33**. *If $(\mathcal{P}^{eo}, \mathcal{R}^{eo})$ is a tame Fredholm pair and $\mathcal{U}^{eo}$ has tame Sobolev order <1, and satisfies $\mathcal{T}^{eo}\mathcal{U}^{eo} = \mathrm{Id} - K^{eo}_1$, $\mathcal{U}^{eo}\mathcal{T}^{eo} = \mathrm{Id} - K^{eo}_2$, then*

$$\begin{aligned}\mathrm{Ind}(\eth^{eo}, \mathcal{R}^{eo}) &= \text{R-Ind}(\mathcal{P}^{eo}, \mathcal{R}^{eo}) \\ &= \mathrm{tr}(\mathcal{P}^{eo}K^{eo}_2\mathcal{P}^{eo}) - \mathrm{tr}(\mathcal{R}^{eo}K^{eo}_1\mathcal{R}^{eo}). \quad [38]\end{aligned}$$

The proof is identical to the proof of theorem 7 in ref. 9. With the relative index formula in Eq. **38** and its expression in terms of a difference of traces in hand, the Agranovich–Dynin formula and the Atiyah–Weinstein index formula can easily be generalized to include $\mathrm{Spin}_{\mathbb{C}}$ manifolds with contact, almost complex boundary assuming only that the Levi form is everywhere nondegenerate. I leave the details to the reader.

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